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2004-03-03

Aspects of mathematical modeling (from a mathematician's viewpoint)

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My experience of philosophers on models.

A few months ago two philosophers spoke about models in a workshop I attended. Their contribution was, as far as I could tell, nonsensical. But mathematicians have said and written things I find interesting. Hence, this is really a mathematician's view of modeling.

What is a model?

Model is not a well-defined concept. Indeed, the above-mentioned philosophers did not even try to define what they meant by a model. The term is used in various ways depending on context and author. Also, even when people mean the same models, they may think different characteristics to be important.

To me, a *model* is an *object designed (or chosen) to be simple to study.* A *mathematical model* is an object that is suited for study by mathematical methods. Often the desired object of study is not actually the model but (some portion of) *reality*, in which case the models should be in some way representative of reality. However, models do not need to represent anything. They can be interesting to study in their own right, such as Conway's Game of Life.

Examples of models.

- A globe is a model of the Earth.
- A model of a big construction such as a housing complex is just that.
- Crick and Watson built a model of DNA as a double helix of balls and sticks.
- The billion-letter word in the alphabet ACGT called the Human Genome Sequence is also a model of DNA.
- The ideal pendulum (a point mass connected to a frictionless joint by a weightless, non-elastic arm) is a model of a real pendulum.
- *Homo economicus* (an agent that always optimizes his own good) is a model of man.
- A Turing machine is a model of a computer.
- A computer is a model of the brain.
- Two rational players bargaining under a veil of ignorance is a model for studying the evolution of the social contract .
- A Markov model of which part of speech (noun, pronoun, verb, etc.) usually follows two previous parts is a model of language.
- Nonnegative real numbers model magnitudes.

Why not study the real world directly?

A first answer lies in my definition of a model: It is easier to study a model than the real world. But why is it easier? It might seem to be a complicated detour to construct a model instead of just studying the real world, which is already there. I think part of the answer is that modeling is not only about computational simplification, but also as much about conceptual simplification. The human mind has evolved to think about certain things. In order to think about other things, we must express them metaphorically as things we know how to think about. The human mind is particularly good at thinking about two-dimensional geometry involving certain basic shapes (like point, line, triangle, square, circle), preferably of size comparable to objects humans can handle manually, hence the popularity of such components in models.

How can models be used?

Models serve both for *prediction* and *explanation* of phenomena.

Suppose a meteorologist wants to compute a weather forecast. She will do her computations in a model of the world, where the weather is described exactly by a finite number of measurements, and governed exactly by some equations. The closer this model is to the real world, the better the prediction will be.

Suppose a phenomenon is found in the real world, such as the emergence of a black market for rental contracts on apartments in central Stockholm. In order to find an explanation, we can model people as *Homo economicus* and the set of rental apartments in Stockholm as a collection of indivisible goods with prices given by rent regulation. Finally, assume that most people prefer central locations to peripheral locations. If in the analysis of such a model one finds that many people will be willing to pay a higher price than the regulated price to secure a central location, then we have an explanation of why a black market would emerge *in the model*. This supports a similar explanation of the phenomenon in the real world, although there is still a possibility that some other explanatory principle is more important (such as there being a cultural tradition of paying previous owners for yielding their property to you, spreading into the realm of rental contracts where it should not apply).

Knauer's questions

Ulrich Knauer, *Philosophical aspects of mathematical modeling,* Proc. of 2nd Int. Conf. on Math. Modeling (1980), 739-743.

In his paper on mathematical modeling, Knauer asks three fundamental questions:

1. Why is it that mathematics (a product of the human mind) can be used to describe reality?

2. In which ways does this happen?

3. What are the advantages of mathematical descriptions?

Suggested answers:

1. This phenomenon is called "The unreasonable effectiveness of mathematics" after the title of a paper by the great physicist Eugene Wigner (1960). Recommended reading is also a paper with the same title by the mathematician Hamming (1980) and a more recent paper by the computer scientist Jef Raskin (1998). One part of the answer (cf. Knauer) is that humans have designed mathematics in order to describe reality. Another part of the answer (cf. Raskin) is that the human mind has evolved to be able to grasp important aspects of reality,

including an innate sense of logic. (One can also speculate about why reality is describable at all.)

2. Mathematical structures (points, lines, numbers, graphs, etc.) model actual structures; equations and algorithms model actual governing principles.

3. Mathematical descriptions have two main advantages. First, they make possible computations, simulations and generally logical reasoning. Second, in order to make a mathematical model the modeler is forced to make assumptions *precise*; also, the questions to which answers are obtained from studying the model will be precise.

Which parts of the world are suited for mathematical modeling?

Suitable for mathematical modeling are systems where actors/objects are expected to behave in some kind of law-abiding manner, and where the relevant entities can be measured in a well-defined way.

As a consequence, we have much more successful models in physics (in particular on human scale, but even at micro level and cosmic level) than in life sciences. Success here refers to things like predictive power. For instance, we can make more accurate predictions about the position of the planets in the solar system a million years from now than about our own feelings in an hour from now.

What good are mathematical models of systems not suited for such modeling?

As long as the models are well-defined they may still serve a purpose for predictions and explanations. However, there are at least three common problems in this area.

1. Since the simplifications involved will be drastic, more care must be taken in drawing conclusions about the real world from the model. Unfortunately, since mathematics itself has a convincing quality, its presence seems to invite this kind of abuse.
2. In order to obtain a well-defined model it is tempting to discard aspects of the real problem not because they are unimportant but because they are difficult to define. This will lead to models that are successful as long as one does not think about the difficult aspects of the problem, which may lead people away from thinking about these aspects although they are important.
3. In order to obtain a model that is not drastically simplified, the modeler makes the model so complex (in number of parameters and assumptions about their relations) that clarity is lost. The lack of transparency makes it difficult to check how robust the conclusions are with respect to data and assumptions.

Example of a model with problem of type 1.

In a recent public debate on sexual discrimination in Swedish universities, Bo Rothstein pointed out that a model used by University Chancellor Sigbrit Franke had problems of type 1. The model assumed that people's experience of being discriminated against can be captured as a vector in $(Z_7)^4$ by a set of four questions in a questionnaire. When drawing conclusions about the real world, Franke then interpreted the single point (0,0,0,0) as "no discrimination" and all other points (2400 in all) as "discrimination", concluding that discrimination was remarkably widespread. Of course, feelings are very badly suited for mathematical modeling, due to the lack of universality that could be a basis for well-defined measures. Thus, computing the share of respondents mapped to (0,0,0,0) is a rather pointless exercise because there is no well-defined interpretation of the result. In other words, when Franke interprets the respondents mapped to other points as discriminated against, this may not be how they would describe themselves. Even less will the picture arising in the minds of people reading about "discrimination" necessarily be in accordance with the reality the model was meant to capture.

Example of a model with problem of type 2.

What is the explanation for the phenomenon of recent years where boards of big companies design bonus systems for top management that give them compensation of celestial proportions even in cases where the companies then turn out to have been gravely mismanaged? One popular explanation says that boards are populated by other CEOs, and they are all rewarding each other on the expense of the companies. However, on many of these boards there have been other kinds of people as well, such as union representatives and politicians, who have had no personal gain of this bonus culture. I believe the explanation also involves a model with problem of type 2, along the following lines.

The board members consider the question *How can we make top management work for the good of the company?* They then make a model: Top management executives are modeled as *Homo economicus.* The good of the company is modeled as increasing stock price. The conclusion is simple: Top management will do whatever it takes to increase their payoff, so pay them according to the development of the stock price. The elegance of the assumptions and the immediate logic of the argument will seem convincing to most board members.

By type 2 problems with this model I mean that important aspects have been left out. To begin with, although the *Homo economicus* model of man is, I think, a good approximation in many economic decision situations, it is not good in other aspects of life. For instance, it is simply false that motivation for work comes mainly from monetary payoffs. Some other important factors are joy, challenge, and a feeling of responsibility toward both abstract values and fellow people. More importantly, an increasing stock price (in the short run!) has no other merit as a model of the "good of the company" than the fact that it is easy to measure. Other more important aspects for the long-term survival and success of the company include management of resources, ideas and staff - but the quality of such things is more difficult to measure. In all, in a more complex world than the model used in the board room, the stock price-based bonus system might well create executives who consciously make decisions that are *worse* for the company than the decisions they would otherwise have made, in order to boost stock price in the short run.

Example of a model with problem of type 3.

Economists Fredrik Bergström and Mikael Sandström published a paper where they claimed to have proved that competition from private schools made public schools perform better (in the sense that grades became higher). Their statistical model tries to explain data about grades in schools ("dependent variables") through data about the pupils, the parents, the schools, the regional geographic and financial conditions, and in particular the share of private schools ("independent variables"). The model is very complex; I have not been able to understand it from reading their paper. It produces results, though, saying that there is a significant positive effect on grades in public schools from competition of private schools.

Another economist, Sören Wibe (in *Ekonomisk Debatt, no 3, 2002*), took the pains of trying to understand the model and reproduce the analysis of Bergström and Sandström. Wibe found that the results were not robust. In fact, the significant effect vanished when he used other data samples, and it also vanished when he changed the set of independent variables.

Bergström and Sandström (also in *Ekonomisk Debatt, no 3, 2002*) viciously replied that Wibe didn't understand their model at all, so that the model he had analyzed was too simple and not the one they had used.

This is a model with problem of type 3; it is so complex that not even another expert can penetrate it. This means that, however convinced the creators of the model are of its accuracy, the model has no explanatory power, for no one can understand how the result follows from the data.

By "no one" I mean also that the creators themselves do not fully understand. In this particular case, there were points in the reply of Bergström and Sandström that revealed a lack of understanding of the workings of statistical models:

- Wibe wrote: "As anyone knows who have worked with statistical analysis, one can always succeed in finding correlations between data sets, even though they actually independent." Bergström and Sandström replied: "As anyone knows who have worked with statistical analysis, one can always succeed in avoiding to find a correlation between data sets, even when there is one." This sounds good but it is just not true, at least not in the same sense as Wibe's converse statement.
- Wibe had found that when he removed one municipality (Pajala) from the data set, the significant effect vanished, and hence the result is not robust. Bergström and Sandström replied that it seemed very arbitrary to remove Pajala; instead they systematically removed all combinations of one, two and tree municipalities and found that the significant effect vanished only for very few of these combinations. For this reason, they concluded the effect to be robust. But this is an absurd conclusion since the point they actually want to make is that there is a causal relationship from competition of private schools to increasing grades in public schools. Such a relationship ought to be visible in any reasonably big sample of municipalities. One counterexample such as the correlation vanishing when Pajala is removed from the sample is sufficient to demolish the argument for a general causality!

Example of a model with problems of type 1, 2 and 3.

In a newspaper piece based on a recent paper by sociologists Guillermina Jasso and Eva M. Meyersson Milgrom on fairness of CEO compensations, Meyersson Milgrom claim that "Swedish MBA students believe 1 821 397 dollars per year is a fair compensation for an American male CEO for a company worth 50 billion dollars, who is 50 years old, with sixteen years of education and five years experience from industry." The exactness of the compensation (seven digits) has a bizarre flavor to an applied mathematician. It sounds like an average value, but it is totally unbelievable that the standard deviation of the different opinions could be in the single dollar range warranting this degree of accuracy.

What Jasso and Meyersson Milgrom have done is the following. They have defined some factors that might influence opinions about fairness of CEO compensation (age, experience, sex, education, size of company). They have then sampled 80 combinations (out of 46 million possibilities) of such factors together with a hypothetical salary, in two decks of forty "vignettes" each. Each respondent of a population of 41 Swedish MBA students looked at one of the two decks, and rated each vignette using a personal, subjective scale, according to how much they thought this CEO was under- or overcompensated. From this data, they estimated the "just rewards" and fitted a linear model.

Problems, as I see it, with this model are:

In drawing conclusions, the authors claim that Swedish MBA students believe a certain salary to be fair for a certain vignette. In fact, it is not clear whether this vignette was part of the study. If it was, then only about 20 respondents will have seen it. No one has been asked about what salary they think is fair; instead this is inferred from the respondents' personal scale. The claim is not well-founded. (Type 1)

Meyersson writes that their model captures precisely those factors that determine opinions on fair CEO rewards. But they have no factor mirroring the present quality of the work of the CEO, nor what he or she has achieved in the past. To many people, at least to me, those factors are even more important than age or education when it comes to deciding a fair reward. (Type 2)

The model is quite complex, and hence not instantaneously transparent. Since the underlying assumption, that a linear model of these factors (and also some squared parameters) should be applicable in the entire parameter space, is astounding to a mathematician, it is sad that they do not even mention the question of robustness. How good a predictor is their model for the answers of a new respondent rating a new sample of vignettes? (Type 3)

Summary

Mathematics is an extremely powerful way of describing the world, creating temptations to use this power in inappropriate ways. Perhaps mathematicians (who are usually skeptic about the applicability of mathematics outside mathematics) ought to be consulted sometimes.